Diffeomorphisms from finite triangulations and absence of 'local' degrees of freedom

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If the diffeomorphism symmetry of general relativity is fully implemented into a path integral quantum theory, the path integral leads to a partition function which is an invariant of smooth manifolds. We comment on the physical implications of results on the classification of smooth and piecewise-linear 4-manifolds which show that the partition function can already be computed from a triangulation of space-time. Such a triangulation characterizes the topology and the differentiable structure, but is completely unrelated to any physical cut-off. It can be arbitrarily refined without affecting the physical predictions and without increasing the number of degrees of freedom proportionally to the volume. Only refinements at the boundary have a physical significance as long as the experimenters who observe through this boundary, can increase the resolution of their measurements. All these are consequences of the symmetries. The Planck scale cut-off expected in quantum gravity is rather a dynamical effect.

I. INTRODUCTION

The quantum theory of gravity is expected to enjoy very special properties some of which are tentatively stated by the *holographic principle* [1]. For more details, we refer to [2].

(A) Holographic principle. In quantum gravity, classical Lorentzian space-time with matter should emerge in such a way that the number of independent quantum states associated with the light sheets of any surface, is bounded by the exponential of the surface area.

Associating the independent degrees of freedom with 2-surfaces rather than 3-volumes is in outright contrast to the familiar properties of local quantum field theory in a fixed metric background. The following statement [1, 2] can therefore often be found in conjunction with (A).

(B) Breakdown of local quantum field theory (QFT). QFT whose degrees of freedom are associated with the points of space, vastly overestimates the number of degrees of freedom of the true quantum theory of gravity.

Principle (A) essentially implies (B). In this letter, we explain how (B) can already be resolved independently of (A) if the diffeomorphism symmetry of general relativity is implemented into the path integral quantum theory. In order to resolve (B) alone, no new physical postulate is needed.

We first recall that any path integral quantization yields a partition function that is an invariant of smooth manifolds. Results on the classification of smooth and piecewise-linear (PL) manifolds show that it can be computed from a triangulation of space-time which characterizes the topology and the differentiable structure.

Naively, one would assume that the degrees of freedom were associated with the simplices of the triangulation, that their number increased with the number of simplices and that the metric size of the simplices provided a short-distance cut-off. Theorems on the classification of PL-manifolds, however, imply that the triangulation in the bulk can be arbitrarily refined without affecting any physical prediction. The actual degrees of freedom are therefore already determined by a suitable *minimal* triangulation. Only refinements at the boundary are physically relevant as long as the experimenters who make their observations through this boundary, can increase the resolution of their measurements.

Whereas the precise form of the path integral for quantum gravity in $d = 3 + 1$ remains elusive, the spin foam models in $d = 2 + 1$ precisely implement the framework developed here. All properties described so far, already follow from the symmetries and are independent of the particular choice of variables or of the action.

II. PATH INTEGRALS

For simplicity, we consider general relativity without matter. Space-time is a smooth oriented 4-manifold with boundary ∂M. The classical theory [3] is the study of the existence and uniqueness of (smooth) metric tensors g on M that satisfy the Einstein equations subject to suitable boundary conditions. In the first order Hilbert–Palatini formulation, one specifies an $SO(1, 3)$-connection A together with a cotetrad field e rather than a metric tensor. Fixing $A|_{\partial M}$ at the boundary, one can derive first order field equations in the bulk which are equivalent to the Einstein equations provided that the cotetrad is non-degenerate. The theory is invariant under space-time diffeomorphisms $f : M \to M$.

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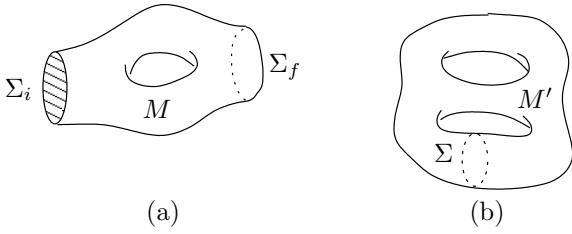FIG. 1: (a) A four-manifold M with boundary $\partial M = \Sigma_i \cup \Sigma_f$. (b) If Σ_i and Σ_f are diffeomorphic, they can be identified to yield the closed manifold M'.

For the path integral, consider a smooth 4-manifold M whose boundary $\partial M = \Sigma_i \cup \Sigma_f$ (Figure 1(a)) is the disjoint union of two smooth 3-manifolds Σ_i and Σ_f to which we associate Hilbert spaces $\mathcal{H}_j$ of 3-geometries, $j = i, f$. These contain suitable wave functionals of connections $A|_{\Sigma_j}$. We denote the connection eigenstates by $\left| A|_{\Sigma_j} \right\rangle$. The path integral,

$$\left\langle A|_{\Sigma_f} \left| T_M \right| A|_{\Sigma_i} \right\rangle = \int_{A|_{\partial M}} \mathcal{D}A\,\mathcal{D}e \, \exp\left(\frac{i}{\hbar} S\right), \quad (1)$$

is the sum over all connections A matching $A|_{\partial M}$, and over all e. It yields the matrix elements of a linear map $T_M \colon \mathcal{H}_i \to \mathcal{H}_f$ between states of 3-geometry. For fixed boundary data, (1) is diffeomorphism invariant in the bulk. If $\Sigma_i \cong \Sigma_f$ are diffeomorphic, we can identify $\Sigma = \Sigma_i = \Sigma_f$ and $\mathcal{H} = \mathcal{H}_i = \mathcal{H}_f$ and consider M' (Figure 1(b)). Provided that the trace over $\mathcal{H}$ can be defined, the *partition function*,

$$Z(M') := \mathrm{tr}_{\mathcal{H}}\, T_M = \int \mathcal{D}A\,\mathcal{D}e \, \exp\left(\frac{i}{\hbar} S\right), \quad (2)$$

where the integral is now unrestricted, is a dimensionless number which depends only on the diffeomorphism class of the smooth manifold M' [4, 5, 6].

Due to the potentially infinite-dimensional Hilbert space $\mathcal{H}$, the partition function (2) may be formally divergent. In $2 + 1$ space-time dimensions, the following examples are known: The Turaev–Viro model [7] (signature $(1, 1, 1)$ and quantized positive cosmological constant) yields finite-dimensional Hilbert spaces and a well-defined $Z(M')$ whereas the Ponzano–Regge model [8] (signature $(1, 1, 1)$ and zero cosmological constant) and the model with Lorentzian signature $(-1, 1, 1)$ [9, 10] have infinite-dimensional Hilbert spaces and divergent $Z(M')$.

III.   COMBINATORIAL TRIANGULATIONS

We assume that we can first focus on the partition function (2) and then learn how to fix A on sub-manifolds in order to go back to (1) which is the actual physically interesting object.

Any $(3 + 1)$-dimensional smooth space-time manifold M is characterized by its *Whitehead triangulation*. This is a purely combinatorial triangulation in terms of k-simplices, $0 \leq k \leq 4$ (vertices, edges, triangles, tetrahedra, 4-simplices), which characterizes the topology and the differentiable structure of M. Combinatorial here means that we specify which triangles are contained in the boundary of which tetrahedra, *etc.*. Under well-known conditions [11], we can glue the simplices according to the combinatorial data and obtain a 4-manifold whose differentiable structure is unique up to diffeomorphism [12, 13]. Conversely, any two diffeomorphic manifolds have equivalent (PL-isomorphic) triangulations [14].

If M is compact with boundary ∂M, the triangulation can be chosen to consist of only a finite number of simplices. The generic quantum geometry measurement involves such a compact M and is given by the generalization of the r.h.s. of (1) to generic boundary [15]. Let $M = [0, 1] \times B^3$ where B^3 is the 3-ball. The boundary ∂M is the union of $\{0\} \times B^3$ (initial preparation of a space-like 3-geometry), $[0, 1] \times S^2$ (time-like geometry which represents a classical clock in the laboratory surrounding the experiment while the system is kept isolated), and $\{1\} \times B^3$ (final measurement of the space-like 3-geometry). Measurements of this type involve both classical observers and a classical clock in the laboratory. They can in principle be defined operationally.

Notice that some authors call other diffeomorphism invariant expressions 'observables', for example, expectation values of scalars with respect to the measure of (2). We do not discuss these global expressions since it is not known whether their measurement can be defined operationally.

Let us sketch when triangulations are equivalent: two finite triangulations are equivalent if and only if they are related by a finite sequence of local modifications, called *Pachner moves* [16], which are elementary shellings for manifolds with boundary and bistellar moves in the bulk. The latter change the triangulation only inside some given polyhedron and replace one possible subdivision of its interior by another one. Figure 2 shows the moves in $d = 3$. The moves in $d = 4$ are illustrated in detail in [17].

Sequences of Pachner moves are the combinatorial equivalent of applying space-time diffeomorphisms. The partition function (2) and the path integral (1) in the bulk therefore have to be invariant under Pachner moves. The same holds, for example, for expectation values of scalars under the measure of (2).

Having understood the moves, we can interpret the role of the triangulation physically. Given the experience with lattice field theory, the naive idea would be to think that the triangulation provided a cut-off by introducing a physical length for the edges of the triangulation. In view of the move of Figure 2(a) which subdivides one simplex into four and which leaves all physical predictions invariant, this cannot be true! The triangulation is rather purely combinatorial. It does not carry any met-
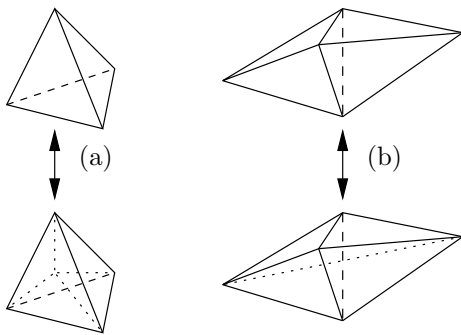
FIG. 2: Pachner moves in $d = 3$. (a) the $1 \leftrightarrow 4$ move subdivides one tetrahedron into four. (b) the $2 \leftrightarrow 3$ move changes the subdivision of a diamond from two tetrahedra to three ones (glued along the dotted line in the bottom picture).
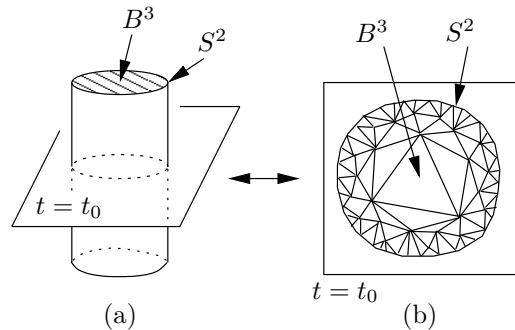

FIG. 3: (a) A tubular region of topology $\mathbb{R} \times B^3$ with boundary $\mathbb{R} \times S^2$ in standard space $\mathbb{R}^4$. (b) A slice at foliation parameter $t = t_0$. The triangulation of the boundary S^2 is as fine as required. In the interior of B^3, Pachner moves have been used to simplify the triangulation.

ric information. The choice of one of several equivalent triangulations is completely analogous to the choice of one of several atlases of coordinate systems for the same smooth manifold. It has no physical significance at all.

The metric data encoded in the path integral (1) and in particular the expected Planck scale cut-off rather have a dynamical origin. The known models in $2+1$ dimensions mentioned above, dynamically assign metric information to the simplices of the triangulation. The Turaev–Viro and the Ponzano–Regge model, for example, endow the 1-simplices (edges) with lengths $(j + 1/2)\ell_P$ where $\ell_P = \hbar G/c^2$ denotes the Planck length and j is a half-integer. In $3 + 1$ dimensions, it is expected that the Planck area $a_P = \hbar G/c^3$ plays the role of the fundamental geometric quantum.

IV. NO LOCAL DEGREES OF FREEDOM

As an application, let us consider a tubular space-time region $M = \mathbb{R} \times B^3 \subseteq \mathbb{R}^4$, Figure 3(a), with boundary $\partial M = \mathbb{R} \times S^2$. Figure 3(b) shows a slice of (a) for some foliation parameter $t = t_0$.

The experiment performed on the space-time region M and measured through ∂M, requires a certain resolution which determines how many simplices are needed in ∂M. Beyond these boundary conditions, we are free to choose any equivalent triangulation in the bulk, in particular a very coarse one, without affecting any physical prediction. Figure 3(b) is deliberately drawn in a very suggestive way. Of course, the apparent size of the simplices in the Figure is completely arbitrary. What matters is only the combinatorics.

Since the path integral dynamically assigns multiples of the Planck length $\ell_P/2$ to the edges in $d = 2 + 1$ or multiples of the Planck area a_P to the triangles in $d = 3 + 1$, there is the following indirect bound on the number of simplices in ∂M that is compatible with a given classical geometry.

Whenever the quantum theory of gravity on M has

a classical limit in which the area of S^2 is A, it makes sense to increase the number n of triangles in S^2 only up to $n \sim A/a_P$. Triangulations with more triangles in ∂M are always assigned larger total areas and their states therefore be suppressed in the very same classical limit.

We have seen that the degrees of freedom are associated with the simplices only up to equivalence of triangulations. In particular, their number does not increase with the volume of the spatial region B^3 (consider t as 'time'). The minimum number of simplices is rather determined by the boundary conditions we impose on S^2, and the number of simplices in the boundary is naturally bounded by A/a_P whenever there emerges a classical geometry such that S^2 has a total area of A.

Standard QFT, in contrast, essentially assigns harmonic oscillators, each with an infinite-dimensional Hilbert space, to the uncountably many points of space. We have reduced this to Hilbert spaces associated with only a finite number of simplices. The major conceptual difference is that standard QFT constructs Hilbert spaces for the fields *relative* to a fixed Riemannian 3-manifold, whereas the Hilbert spaces of (1) represent equivalence classes of 3-geometries up to spatial diffeomorphisms.

We note that none of our arguments changes substantially if we add matter or gauge fields or if we formulate the classical theory in terms of other variables. Recall that the symmetry under space-time diffeomorphisms is not special to general relativity, but shared by many other classical field theories, indicating a potential impact on particle physics in general. Further details on path integral quantization of general relativity and invariants of smooth manifolds will be presented in [18].

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